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Parabolic Cylindrical Description of Velocity and Acceleration Tensor in Bipolar Coordinate Using Tensor Analysis

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ABSTRACT

Instantaneous velocity and acceleration are often studied and expressed in Cartesian, Circular Cylindrical, and Spherical Coordinate systems but it is a well-known fact that some bodies cannot be perfectly described in these coordinate systems, so they required some other curvilinear systems such as oblate spheroidal, parabolic cylindrical, elliptic cylindrical bipolar, and others. It is of important interest in theoretical physics to establish equations of motion in Bipolar Coordinate system, which is essentially suitable to accurately describe the motion of bipolar bodies such as hyperbolas, ellipses and other curves in the universe. In this work, we derived a new expression for the instantaneous velocity and acceleration vector of bodies and test particles in the bipolar coordinate system using tensor analysis. The Laplacian of a scalar field in this bipolar coordinate system was also derived which has applications in mechanics.

INTRODUCTION

According to Howusu, (2003), in Newton Mechanics, motions of bipolar bodies are treated with the assumption that they are perfectly cylindrical or circular. Howusu, (2003) also asserted that Bipolar Coordinates are two-dimensional orthogonal coordinate systems and that there are two commonly defined types of bipolar coordinates. The first is based on the apollonian circles. The curves of constant ξ and η are circles that intersect at right angles. The coordinates have two foci F_1 and F_2 which are generally taken to be fixed at (-a, 0) and (a, 0), respectively, on the x-axis of a Cartesian coordinates system. The second system is two-center bipolar coordinates. There is also a third coordinate system that is based on two poles (biangular coordinates)(Howusu, 2003). For instance: The classic applications of bipolar coordinates are in solving partial differential equations like Laplace's equation or the Helmholtz equation, for which bipolar coordinates allow the separation of variables. A typical example would be the electrical field surrounding two parallel cylindrical conductors. Hence, the Laplacian is given by (Howusu, 2003)

$$\nabla^2\emptyset = \frac{1}{a^2}(cosh\eta - cos\xi)^2\left(\frac{\partial^2\emptyset}{\partial\xi^2} + \frac{\partial^2\emptyset}{\partial\eta^2}\right) \quad (1)$$

Verily, these approximations are adopted for essential simplicity and convenience. The real fact of nature is that all bipolar bodies are sometimes used to describe other curves having two singular points (foci), such as ellipses, hyperbolas, and Cassini ovals. However, Omonile et al, 2014a, asserted that the term bipolar coordinates are reserved for the coordinates described here and are never used to describe coordinates associated with those other curves in the universe, such as elliptic coordinate. It is expedient that their bipolar geometry has corresponding consequences in the motion of all the particles in their gravitational field. These effects will exist both in Newtonian mechanics and in Einstein's theory. Consequently, a way of the solution has been provided for the solution of the equation of motion of bipolar bodies by deriving the equations of motion for bipolar coordinates using vector analysis (Omonile et al, 2014a). Herein, we shall introduce the solution to bipolar motion using a more generalized concept of tensor analysis. Hence, in this work, we shall approach the solution to the motion (velocity and acceleration) of these bodies using the tensor analysis in bipolar coordinates (Omonile et al, 2014b).

THEORY

The scale factors of the bipolar coordinates are defined as:

$$h\varepsilon = \frac{a}{\cosh\eta - \cos\varepsilon} \tag{2}$$

$$h\eta = \frac{a}{\cosh\eta - \cos\varepsilon} \tag{3}$$

$$hz = hz \tag{4}$$

These metric tensors give the unit vector, line elements, as well as gradients, divergence, curl, and Laplacian operators in Bipolar coordinates, according to the theory of orthogonal curvilinear coordinates. These quantities are necessary and sufficient for the derivation of the fields of all bipolar distributions of mass, charge, and current. Now for the derivations of motion for test particles in these fields, we shall derive an expression for instantaneous velocity and acceleration in bipolar coordinates (Howusu, 2003).

The metric tensors for bipolar coordinates are expressed as:

$$g_{00} = 1 \tag{5}$$

$$g_{11} = \frac{a^2}{(\cosh\eta - \cos\varepsilon)^2} \tag{6}$$

$$g_{22} = \frac{a^2}{(\cosh\eta - \cos\varepsilon)^2} \tag{7}$$

$$g_{33} = 1 \tag{8}$$

$$g_{00} = 0;\ \text{otherwise} \tag{9}$$

Hence, the contra-variant metric tensors are given as:

$$g^{00} = 1 \tag{10}$$

$$g^{11} = \frac{(cosh\eta - cos\xi)^2}{a^2} \quad (11)$$

$$g^{22} = \frac{(cosh\eta - cos\xi)^2}{a^2} \quad (12)$$

$$g^{33} = 1 \quad (13)$$

$g^{uv} = 0$; otherwise (14)

Velocity Tensor
The velocity tensor component in bipolar coordinate is given as (Omonile et al, 2014c):

$$\bar{U} = u_{\xi}\hat{\xi} + u_{\eta}\hat{\eta} + u_z\hat{z} \quad (15)$$

Given that (16)

$$u_{\xi} = (g_{11})^{\frac{1}{2}} \quad (17)$$

$$u_{\eta} = (g_{22})^{\frac{1}{2}} \quad (18)$$

$$u_z = (g_{33})^{\frac{1}{2}} \quad (19)$$

$$\bar{U} = \left(\frac{a^2}{cosh\eta - cos\xi}\right)^{\frac{1}{2}} \dot{\xi}\,\hat{\xi} + \left(\frac{a^2}{cosh\eta - cos\xi}\right)^{\frac{1}{2}} \frac{1}{2}\,\dot{\eta}\,\hat{\eta} + \dot{z}\,\hat{z} \quad (20)$$

This is the expression for velocity tensor in bipolar coordinate.

ACCELERATION TENSOR IN BIPOLAR COORDINATE
Acceleration tensor components in bipolar coordinates are expressed as:

$$\bar{a} = a^0 + a^1 + a^2 + a^3 \quad (21)$$

According to the theory of tensor analysis, the acceleration tensor is given as (Spiegel,1974).

$$a^{\propto} = \ddot{x}^{\propto} + \Gamma^{\alpha}_{uv}\dot{x}^u\dot{x}^v \quad (22)$$

where, $\Gamma^{\alpha}_{uv} = \frac{1}{2}\, g^{u\epsilon}\,(g_{v\epsilon,u} + g_{\epsilon u,v} - g_{vu,\epsilon})$

(23)

$\in = 0,1,2,3$; counters (24)

Γ^{α}_{uv} = Christoffel symbol or coefficient of affine connection

Note:

$$g^{u\in} = \begin{cases} false; u \neq \in \\ true\ u = \in \end{cases} \quad (25)$$

Where: $x^0 = 1$; $x^1 = \varepsilon$; $x^2 = \eta$; $x^3 = z$

By the application of equation (23), the non-zero values are obtained to be:

$$\Gamma^{11}_{00} = 0 \quad (26)$$

$$\Gamma^{1}_{01} = 0 \quad (27)$$

$$\Gamma^{1}_{11} = \frac{-sin\varepsilon}{cosh\eta - cos\varepsilon} \quad (28)$$

$$\Gamma^{1}_{12} = \Gamma^{1}_{21} = \frac{-sin\varepsilon}{(cosh\eta - cos\varepsilon)} \quad (29)$$

$$\Gamma^{1}_{13} = \Gamma^{1}_{31} = 0 \quad (30)$$

$$\Gamma^{1}_{22} = \frac{sin\varepsilon}{(cosh\eta - cos\varepsilon)} \quad (31)$$

$$\Gamma^{1}_{23} = \Gamma^{1}_{32} = 0 \quad (32)$$

$$\Gamma^{1}_{33} = 0 \quad (33)$$

$$\Gamma^{2}_{11} = \frac{sinh\eta}{(cosh\eta - cos\varepsilon)} \quad (34)$$

$$\Gamma^{2}_{12} = \Gamma^{2}_{21} = \frac{-sin\varepsilon}{(cosh\eta - cos\varepsilon)} \quad (35)$$

$$\Gamma^{2}_{22} = \frac{-sinh\eta}{(cosh\eta - cos\varepsilon)} \quad (36)$$

$$\Gamma^{2}_{23} = \Gamma^{2}_{32} = 0 \quad (37)$$

$$\Gamma^{2}_{13} = \Gamma^{2}_{31} = 0 \quad (38)$$

$$\Gamma^{2}_{33} = 0 \quad (39)$$

$$\Gamma^{3}_{11} = 0 \quad (40)$$

$$\Gamma^{3}_{12} = \Gamma^{3}_{21} = 0 \quad (41)$$

$$\Gamma^{3}_{22} = 0 \quad (42)$$

$$\Gamma^{3}_{23} = \Gamma^{3}_{32} = 0 \quad (43)$$

$$\Gamma^{3}_{33} = 0 \quad (44)$$

$\Gamma_{13}^{3} = \Gamma_{31}^{3} = 0$ (45)

$\Gamma_{uv}^{\alpha} = 0$, otherwise, (46)

Note: $\Gamma_{uv}^{\alpha} = \Gamma_{vu}^{\alpha}$ (47)

It follows from the above results obtained that:

$a^0 = \ddot{x}^0 + \Gamma_{uv}^{0} = \dot{x}^u \dot{x}^v$ (48)

$a^0 = 0$ (49)

$a_{\xi}^{1} = \ddot{x}^1 + \Gamma_{uv}^{1} = \dot{x}^u \dot{x}^v$ (50)

$a_{\xi}^{1} = \left\{\ddot{\xi} - \frac{\sin\xi}{\cosh\eta - \cos\xi}(\dot{\xi})^2 - \frac{2\sinh\eta}{\cosh\eta - \cos\xi}\dot{\xi}\dot{\eta} + \frac{\sin\xi}{\cosh\eta - \cos\xi}(\dot{\eta})^2\right\}$ (51)

$a_{\eta}^{2} = \ddot{x}^2 + \Gamma_{uv}^{2} = \dot{x}^u \dot{x}^v$ (52)

$a_{\eta}^{1} = \left\{\ddot{\eta} + \frac{\sinh\eta}{\cosh\eta - \cos\xi}(\dot{\xi})^2 - \frac{2\sin\xi}{\cosh\eta - \cos\xi}\dot{\xi}\dot{\eta} - \frac{\sinh\eta}{\cosh\eta - \cos\xi}(\dot{\eta})^2\right\}$ (53)

$a_{z}^{3} = \ddot{x}^3 + \Gamma_{uv}^{3} = \dot{x}^u \dot{x}^v$ (54)

$a_{z}^{3} = \ddot{x}^3 + 0$ (55)

$a_{z}^{3} = \ddot{x}^3$ (56)

$a_{z}^{3} = \ddot{z}$ (57)

The above results defined the desired acceleration tensor in Bipolar coordinates. Hence, the acceleration vector is defined as:

$\underline{a} = [a_{\xi}, a_{\eta}, a_z]$ (58)

or

$\underline{a} = a_{\xi}\hat{\xi} + a_{\eta}\hat{\eta} + a_z\hat{z}$

Where:

$a_{\xi} = (g_{11})^{\frac{1}{2}} a^1$ (59)

$a_{\eta} = (g_{22})^{\frac{1}{2}} a^2$ (60)

$a_{z} = (g_{33})^{\frac{1}{2}} a^3$ (61)

Hence, we defined the acceleration vector as follows:

$$a_{\xi} = \left[\frac{a^2}{(cosh\eta - cos\xi)^2}\right]^{\frac{1}{2}} \left[\ddot{\xi} - \frac{sin\xi}{cosh\eta} (\dot{\xi})^2 - \frac{2sinh\eta}{cosh\eta - cos\xi} \dot{\xi}\dot{\eta} + \frac{sin\xi}{cosh\eta - cos\xi} (\dot{\eta})^2\right] \tag{62}$$

$$a_{\eta} = \left[\frac{a^2}{(cosh\eta - cos\xi)^2}\right]^{\frac{1}{2}} \left[\ddot{\eta} - \frac{sinh\eta}{(cosh\eta - cos\xi)} (\dot{\xi})^2 - \frac{2sin\xi}{cosh\eta - cos\xi} \dot{\xi}\dot{\eta} + \frac{sinh\eta}{cosh\eta - cos\xi} (\dot{\eta})^2\right] \tag{63}$$

$$a_z = \ddot{a} \tag{64}$$

The expression is the acceleration vector in bipolar coordinate.

RESULTS AND DISCUSSION

In this paper, we derived the component of velocity as (15) - (20) and acceleration as (58) - (64) in a bipolar coordinates system using tensor analysis. The outcome obtained in this work are essential and appropriate for expressing all mechanical magnitudes (linear momentum, kinetic energy, Lagrangian and Hamiltonian) in terms of bipolar coordinate systems. Also, the results of this work have paved the way for expressing all dynamic laws of motion (Newton's laws, Lagrange's law, Hamiltonian's law, Einstein's Special Relativistic Law of Motion, and Schrödinger Law of Quantum Mechanics) entirely in bipolar coordinate systems.

CONCLUSION

This work provides the desired result suitable in the theory of three-dimensional orthogonal coordinates system with a more dynamic approach to tensor analysis. This method is applicable in other familiar systems such as the Cartesian, oblate spheroidal coordinate, circular cylindrical system, and others.

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